The Philosophy of Numbers

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May 8 2024

Introduction

In modern opinion, Math and Philosophy are two completely different subjects. Most students would be baffled to open a philosophy textbook and find an integral, or to be asked the meaning of life on a geometry quiz. The two subjects are taught with almost no similarity, and many even consider Mathematics to be its own separate language. Over the last few centuries cutting edge math has become increasingly specialized and less approachable for average academics. As the focus has shifted to computation and theory, the origins of mathematical reasoning have fallen from the spotlight, and its rich philosophical influence has been forgotten.

Despite being seen as completely separate fields, philosophy was an inseparable part of mathematics for the majority of history. This paper will discuss three foundational mathematicians from three different time periods, and discuss the link between their contributions to mathematics and the philosophy of their era.

1 Ancient Mathematics - Euclid

Euclid was a Greek mathematician living in Alexandria around 300 BC (Katz, 1998). He is often referred to as the "Father of Geometry" for his work: Elements, which was a comprehensive collection of geometric knowledge at the time.

1.1 Philosophy

While Euclid was not a philosopher, he was heavily influenced by Aristotle–a Greek philosopher who lived around the same time. Aristotle was a student of Plato and Socrates, and is widely considered the founder of "deductive reasoning" (Shields). Deductive reasoning is the practice of establishing agreed truths, then using these and logical arguments to prove other truths. The chain extends, and we are able to create a whole set of common beliefs. Translated into mathematical language: deductive reasoning establishes axioms, and uses these to prove theories.

1.2 Elements

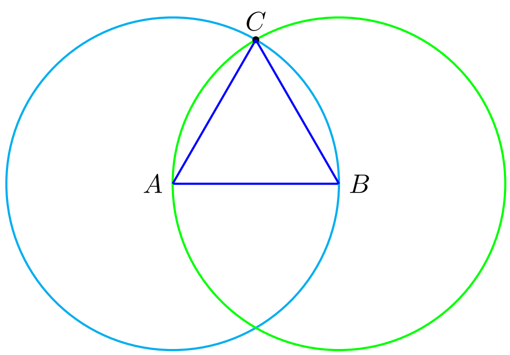
Euclid's *Elements* is famous for applying this method of deductive reasoning to mathematics. In Book I of the *Elements*, he proposes 5 postulates (or axioms) that he establishes as true, and uses to prove propositions (theories) throughout the entire book. For example:

Postulate 1: "To draw a straight line from any point to any point" (Joyce)

The postulates can be compared to philosophical truths the reader agrees to accept, and as Euclid proves each proposition, the set of truths extends. Below is a proof of Book I, Proposition I similar to the one Euclid used. It relies on nothing but the definitions and Postulates established earlier in the book.

Book I, Proposition I

To construct an equilateral triangle on a given finite straight line. (Joyce)



1. Draw line $\overline{AB}$.
2. With centers A and B, draw circles with radius $\overline{AB}$, intersecting at point C.
3. Draw lines $\overline{AC}$ and $\overline{BC}$.
4. Observe that $\overline{AC} = \overline{AB}$ because both are radii of the circle centered at A, and similarly $\overline{BC} = \overline{BA}$ as radii of the circle centered at B.
5. Since $\overline{AC} = \overline{AB}$ and $\overline{BC} = \overline{AB}$, it follows that $\overline{AC} = \overline{BC}$.
6. Thus, $\triangle ABC$ is equilateral, having all sides equal.

1.3 Deductive Reasoning

In this way, Euclid is using deductive reasoning to prove mathematical concepts based on a set of agreed axioms. And a critical feature of deductive reasoning: when each proposition is proved, it becomes a truth that can be used later. Using this method Euclid establishes 13 books of mathematical truths on various geometric concepts and distributes them to the Greek world. The Greeks had serious respect for philosophy, so the method of deductive reasoning was seen as highly logical, and adopted as the standard for mathematical reasoning. (Euclid, Elements)

1.4 Legacy

The use of axioms and deductive reasoning formalized the structure of mathematical texts, and actually encouraged the development of new math. It allowed for new math to be systematically proposed and proven in a way mutually agreed upon in the mathematical world. While initially used only for geometry, this method was eventually applied to fields such as geometry, calculus, and statistics. Mathematical proofs still use this method today, however the body of axioms we consider to be true has increased astronomically.

While not a philosopher, it is clear that Euclid's mathematical methods were highly connected with Aristotle's philosophical methods. Known for using similar reasoning on questions of nature and the cosmos, Aristotle is also considered one of the founders of "Natural Philosophy"–which became simply "science" in the 19th Century, and an early pioneer of Rationalist thought. Therefore, while we see a key division between his study of philosophy and questions of physical science, the Greeks would have viewed them as essentially the same thing. To them, science and math were an extension of philosophy–one that used rationality and logic to prove truths about the physical world.

The standard format of mathematical reasoning we have today emerged out of philosophical thinking from the Greeks. For them, philosophy was almost the medium which they used to do math, but today it has just become ingrained in the logic we use to solve problems and make arguments. Math's origins in argumentative logic allowed math to grow beyond the Greeks' wildest expectations and become the growing system it is today.

2 Renaissance Math - Descartes

The historical link between philosophy and math is so inseparable that famous mathematicians were often philosophers as well. Rene Descartes is one of the most famous examples of this, making dual contributions to philosophy and mathematics. Descartes was born in France in 1596–a pivotal time in which academics began to rapidly expand throughout Europe. By 1600, there were about 73 Universities in Europe, Greek logic was widely accepted, and academic journals were starting to be published (Emerson, Protocaculus). The majority of mathematical contributions during the Renaissance came from elite scholars like Descartes, who were wealthy, highly educated individuals with expertise across multiple fields.

2.1 Discourse on Method

Descartes published a large body of philosophical work, and is perhaps best known for his 1637 book: *Discourse on Method*. *Discourse* begins by introducing systematic doubt of all things, and seeks building a new foundation of certain knowledge. In his famous statement: *Cogito, ergo sum*–"I think, therefore I am", Descartes establishes his first certain truth and builds upon this to create his philosophical worldview (Hatfield). This statement is indistinguishable from an axiom used in mathematical logic. He then discusses his "Four Rules of Method", which are guiding principles on how to establish truths and solve problems.

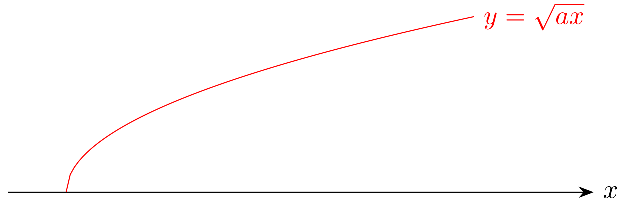
His axiomatic logic is a clear example of the same deductive reasoning used by Euclid and Aristotle, and has clear implications for mathematical problem solving. The connection to math is so apparent, that Descartes adds an appendix to *Discourse* solving geometry problems–and inadvertently creates a new field of mathematics.

2.2 La Geometrie

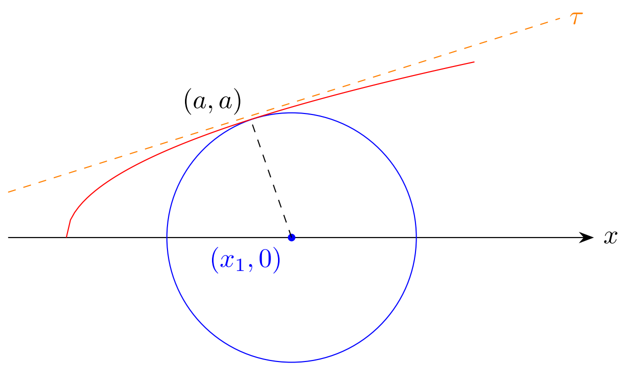
Descartes attaches *La Geometrie* as an appendix to *Discourse* to showcase the application of his philosophical and methodological principles to the field of geometry. Following the Rules of Method established in *Discourse*, Descartes starts with basic geometric principles and uses deductive reasoning to solve more complex geometric theorems (Hatfield). Through a direct application of his methodological emphasis on combining and simplifying problems to make them more solvable, Descartes demonstrates how algebraic equations can be used to describe curves–unifying two previously separate systems and founding the new mathematical field of Analytic Geometry (Emerson, Analytic Geometry). He also creates the x-axis (though slightly different than our modern understanding) and pushes mathematics toward the creation of calculus in the next generation.

He proves a number of important results, and his work on tangent lines is especially important in creating the foundation for what would become calculus. To find the tangent line to a curve, Descartes uses the fact that the tangent line to a circle is perpendicular to the radius.

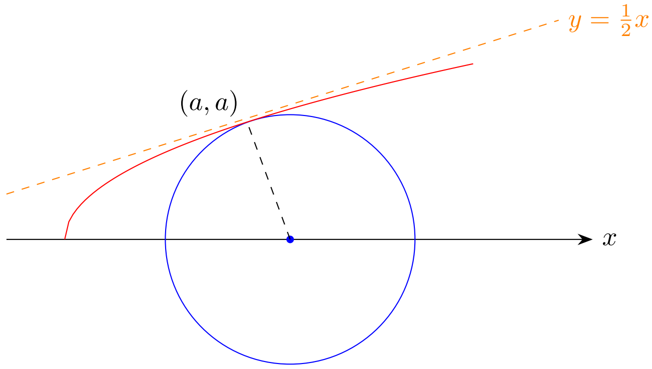
Figure 1 To demonstrate this, we begin by drawing a curve $y = \sqrt{ax}$ on the x axis



2 Our goal is to find the tangent line: τ, at some point: (a, a). To do this we draw a circle at $(x_1, 0)$ with radius r that is perpendicular to the curve y where it intersects the circle.



3 Using a number of algebraic manipulations we set up an equality for the intersection and solve for x_1 and r. This gives us the slope of the tangent line. In this case, the tangent line is $y = \frac{1}{2}$



We can check this using calculus: $f'(x) = \frac{\sqrt{a}}{2\sqrt{x}}$
And plugging in a we find that $f'(a) = \frac{1}{2}$

2.3 Legacy

As we can see, Descartes' contributions to mathematics were incredibly influential. He established that curves can be represented by equations, created the Cartesian coordinate plane we still use today, and his work in Analytical Geometry laid the foundation for the creation of Calculus by Leibniz and Newton (Emerson Analytic Geometry). Most importantly, each of these results were a direct result of his philosophical work.

Descartes' math was an extension of his philosophical methods. His rationalist approach in *Discourse* empowered him to make mathematical revelations that other mathematicians could not, therefore his philosophical revelations were essential for the development of his math. Despite living almost 2000 years after Euclid, his dependence on deductive reasoning shows that mathematical proofs were still based in philosophical methods.

To Descartes, separating the subjects of mathematics and philosophy would have seemed ridiculous. He used philosophy to develop and extend the set of mathematical truths in the same way Euclid used axioms and deductive reasoning to formalize the system of geometry. Just 300 years ago, the two subjects were so similar that one of the most influential rationalist philosophy works and most cutting edge geometry proofs were a part of the same book.

3 Modern Math - Bertrand Russell

In the early 20th century, Bertrand Russell, a philosopher and logician, made profound contributions that continued to demonstrate the inseparability of philosophy and mathematics. During a period marked by the formalization of mathematical theories, Russell's work, particularly in set theory and logic, highlighted foundational issues that bridged philosophical inquiry and mathematical practice.

3.1 Logicism

Russell was a central figure in the development of Logicism, the theory that mathematics is in its essence purely logical and that all mathematical truths are truths of logic. Russell aimed to demonstrate that "all of mathematics was nothing more than higher logic" (Irvine). This perspective is an extension of deductive logic used by previous mathematicians and philosophers like Euclid and Descartes. Russell proposed that all mathematical principles could be derived from logical axioms, and engaged with naive set theory of the time to uncover mathematical truths.

3.2 Russell Paradox

Logical paradoxes have existed for thousands of years, and raise intriguing questions about the dynamics of set theory. One of the most famous and simple of these logical paradoxes is: "This statement is false"

To engage with the flaws in set theory, Russell proposed the following paradox: "In a certain town there is one barber. Every man shaves himself or is shaved by the barber. Who shaves the barber?" This paradox can be generalized to create the Russell set. (Emerson, "Logic and Gödel's Theorem")

The paradox arises within naive set theory by considering the set of all sets that do not contain themselves. Let us define the set R as follows:

$$R = \{S : S \notin S\}$$

The paradox occurs when we question whether R contains itself or not.

Proof by Contradiction

Assume for contradiction that R is a well-defined set. We consider two cases:

1. **Case 1:** $R \in R$
   Suppose $R \in R$. According to the definition of R, if $R \in R$, then R must not contain itself, i.e., $R \notin R$. This is a contradiction.

2. **Case 2:** $R \notin R$
   Suppose $R \notin R$. According to the definition of R, if $R \notin R$, then R must contain itself, i.e., $R \in R$. This too leads to a contradiction.

Since both assumptions lead to contradictions, we conclude that the set R cannot exist under the rules of naive set theory.

This paradox illustrates a fundamental problem in naive set theory, and by pointing it out Russell prompted mathematicians to refine and reformulate the foundations of set theory to avoid such paradoxes. This criticism led to the development of more robust axiomatic set theories, such as Peano's axioms of arithmetic and Zermelo-Fraenkel Set theory (Emerson, "Logic and Gödel's Theorem"). These theories impose stricter conditions on set formation, which prevent the formulation of sets like the Russell set through careful restrictions on the kinds of properties that can be used to define sets

Russell's approach to this paradox and all his mathematical work was distinctly philosophical. In his "Principia Mathematica," co-authored with Alfred North Whitehead, and "Introduction to Mathematical Philosophy," he applied rigorous logical analysis to establish a foundation of truth for mathematics. In these texts Russell broke down traditional concepts of math into logical components and rebuilt them to solve problems using rigorous logic and formal axioms (Irvine 2009) Similar to Descartes–his uncompromising philosophical methods were the driving force behind his mathematical contributions.

3.3 Legacy and Continued Relevance

Russell's efforts to formalize mathematics through logical principles highlights the essential role of philosophical reasoning in mathematical thought. His work has lasting implications for the philosophy of mathematics, demonstrating that even modern mathematical practices are deeply embedded with philosophical questions about truth, proof, and logical consistency. Set theory itself needed a strong foundation in logic to develop as it is frequently tested using logical contradictions and philosophical proofs (Irvine, 2009). Bertrand Russell's fusion of philosophy and mathematics illustrates the continued intersection of these fields, and once again demonstrates how philosophy is essential to the development of many mathematical practices.

Conclusion

Modern math was born as an extension of philosophy, and for the vast majority of mathematical history the two fields were linked. This essay has explored how Euclid, Descartes, and Russell each engage with both philosophy and mathematics.

The use of deductive reasoning and axioms in ancient Greece formalized mathematical proofs and allowed new mathematicians to make meaningful and coherent contributions. Rationalist philosophy allowed Descartes to invent the coordinate plane we use today and redefine the understanding of curves, paving the way for calculus to emerge. Finally, Bertrand Russell used philosophical reasoning to refine modern set theory and encourage the creation of formal mathematical axioms.

All these contributions highlight an essential narrative: the development of math is deeply intertwined with philosophical exploration and reasoning. These thinkers utilized philosophical methods not only to advance mathematical theory, but also to challenge and refine the ways in which we understand mathematical truths.

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